

# MULTIOBJECTIVE OPTIMAL STRUCTURAL CONTROL OF THE NOTRE DAME BUILDING MODEL BENCHMARK<sup>a</sup>

E. A. JOHNSON<sup>\*,†,‡</sup>, P. G. VOULGARIS<sup>§</sup> AND L. A. BERGMAN<sup>¶</sup>

*Department of Aeronautical and Astronautical Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.*

## SUMMARY

Reduced-order, multiobjective optimal controllers are developed for the Notre Dame structural control building model benchmark. Standard  $H_2$ /LQG optimal control excels at noise and disturbance rejection, but may have difficulty with actuator saturation and plant uncertainty. The benchmark problem is adapted to a multiobjective optimal control framework, using  $l_1$  and  $H_\infty$  constraints to improve controller performance, especially attempting to reduce peak responses, avoid saturation, and improve robustness to unmodelled dynamics. The tradeoffs between  $H_2$  performance, output peak magnitudes, and robust stability are examined. Several optimal controllers and their performance on the benchmark are given. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: active structural control; earthquake response; multiobjective optimal control;  $H_\infty$  control;  $l_1$  control

## INTRODUCTION

Spencer *et al.*<sup>1,2</sup> detailed a building model benchmark problem for studying active structural control strategies. The problem defines a base plant that models the structural behaviour of a three-story, single-bay, scale model of a building,<sup>3</sup> as well as the dynamics of the sensors (an LVDT, several accelerometers, and a force transducer) and the actuator (an active mass driver or an active tendon member). Such a benchmark problem is invaluable for studying various control strategies with realistic limitations such as actuator saturation, limited actuator stroke, digital controller implementation, reduced-order controllers, and multiple measures of controller performance and robustness.

Standard  $H_2$ /LQG optimal control design methods typically excel at noise and disturbance rejection, but may have difficulty with actuator saturation and plant uncertainty. The addition of  $l_1$  constraints may help accommodate requirements such as limiting control output or other peak measures of system performance. Furthermore,  $H_\infty$  robust control design methods are advantageous since the model of a plant always contains some measure of uncertainty for any real-world application, whether due to modelling errors or long-term changes in system dynamics. With a small sacrifice in the system  $H_2$  norm from that found using an  $H_2$  (LQG) optimal controller, giving slightly larger root mean square response to white noise excitation,

<sup>a</sup> This paper is dedicated to Franz Ziegler, Professor of Rational Mechanics and Head of the *Institut für Allgemeine Mechanik* at the Vienna University of Technology, on the occasion of his 60th birthday

<sup>\*</sup> Correspondence to: E. A. Johnson, Department of Civil Engineering and Geological Sciences, University of Notre Dame, Notre Dame, IN 46556-0767, U.S.A. E-mail: johnsone@nd.edu

<sup>†</sup> Current address: Department of Civil Engineering and Geological Sciences, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

<sup>‡</sup> Formerly, Graduate Research Assistant; currently, Visiting Research Assistant Professor

<sup>§</sup> Associate Professor

<sup>¶</sup> Professor

the  $l_1$  and  $H_\infty$  norms can often be significantly reduced, leading to better performance and more robust stability. The use of all three of these norms to devise an optimal controller can combine the strengths of each.

The general formulation of the mixed  $H_2/H_\infty$  optimal control design problem has been explored in a number of papers (e.g. References 4–6), building on  $H_2$  and  $H_\infty$  control methods in Doyle *et al.*<sup>7–9</sup> The  $l_1$  optimal control problem has also been detailed in various papers and texts (e.g. References 10–12). A number of solution methods have been developed to solve certain mixed-norm control design problems (e.g. References 13 and 14). For example, the homotopy algorithm of Whorton *et al.*<sup>13</sup> uses a linear combination of the objective  $H_2$  and  $H_\infty$  norms, varying the ratio of the linear coefficients and searching incrementally in the  $H_2$  and  $H_\infty$  directions to define a family of mixed-norm designs.

To facilitate numerical solutions to general mixed-norm optimal control problems, Jacques *et al.*<sup>15–19</sup> developed a set of numerical algorithms and accompanying MATLAB<sup>®</sup> code to search for fixed-order controllers that minimize an  $H_2$  norm while constraining one or more  $H_2$ ,  $l_1$  (or  $L_1$ ), and  $H_\infty$  norms below some critical values. These algorithms, which can be used for continuous- or discrete-time systems, have been adapted to the current problem with alteration of the associated software.

### BENCHMARK PROBLEM REQUIREMENTS

The base plant<sup>1,2</sup> is described by the block diagram in Figure 1 and by the state equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}\ddot{\mathbf{x}}_g \\ \mathbf{z} &= \mathbf{C}_z\mathbf{x} + \mathbf{D}_zu + \mathbf{F}_z\ddot{\mathbf{x}}_g \\ \mathbf{y} &= \mathbf{C}_y\mathbf{x} + \mathbf{D}_yu + \mathbf{F}_y\ddot{\mathbf{x}}_g + \mathbf{v} \end{aligned} \quad \mathbf{z} = \begin{bmatrix} x_1 & x_2 & x_3 & x_m \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_m \\ \ddot{x}_1^a & \ddot{x}_2^a & \ddot{x}_3^a & \ddot{x}_m^a \end{bmatrix}^T \quad \mathbf{y} = \mathbf{v} + [x_m \ \ddot{x}_1^a \ \ddot{x}_2^a \ \ddot{x}_3^a \ \ddot{x}_m^a \ \ddot{x}_g]^T \quad (1)$$

where  $\mathbf{x}$  is the state vector (in  $\mathfrak{R}^{28}$  for the active mass driver (AMD) system and in  $\mathfrak{R}^{20}$  for the active tendon system (TEN)),  $\mathbf{y}$  is the measurement vector,  $\mathbf{z}$  is a vector of responses to be regulated,  $u$  is the actuator input,  $\ddot{\mathbf{x}}_g$  is the seismically-induced ground acceleration, and  $\mathbf{v}$  is the sensor noise (independent, zero-mean, Gaussian white noise processes). The components of the output measurement  $\mathbf{y}$  and regulated response  $\mathbf{z}$ , the former corrupted by sensor noise, include the displacement  $x_i$  and velocity  $\dot{x}_i$  of the  $i$ th floor relative to the ground, absolute accelerations of the  $i$ th floor  $\ddot{x}_i^a$  and the ground  $\ddot{x}_g$ , relative actuator displacement  $x_m$  and velocity  $\dot{x}_m$ , and either the active mass absolute acceleration  $\ddot{x}_m^a$  or the active tendon force  $f$ . In the active tendon system, the tendon force  $f$  replaces the active mass acceleration  $\ddot{x}_m^a$  in both  $\mathbf{y}$  and  $\mathbf{z}$  in equation (1). Inter-story drifts are defined as  $d_1(t) = x_1(t)$ ,  $d_2(t) = x_2(t) - x_1(t)$ , and  $d_3(t) = x_3(t) - x_2(t)$ .

The coefficient matrices in equation (1) represent the input–output behavior of the building models in the Notre Dame Structural Dynamics and Control/Earthquake Engineering Laboratory (SDC/EEL) up to 100 Hz (the AMD system) and the National Center for Earthquake Engineering Research (NCEER) up to 50 Hz (the TEN system). Both sets of coefficient matrices were distributed in a MATLAB<sup>®</sup> MAT-file as a part of the benchmark problem definition.<sup>20</sup>

The goal, then, is to design a stable, discrete-time controller with sampling period  $T = 1$  ms, subject to several implementation constraints:

- the sensor noise components are Gaussian rectangular pulse processes of 1 ms duration and root mean square magnitude 0.01 V;
- the A/D and D/A converters at the controller input and output have 12-bit precision and a span of  $\pm 3$  V;
- the controller may have no more than 12 states; and
- the resulting actuator response must be within the bounds of the 6 hard constraints,  $C_i \leq 1$ ,  $1 \leq i \leq 6$ , as defined below.

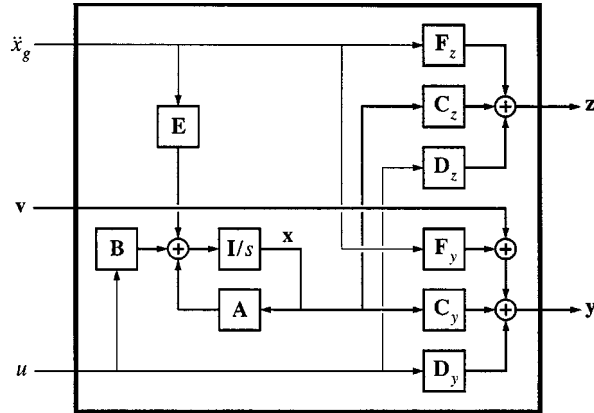


Figure 1. Block diagram of benchmark plant

The performance of the controller is then judged by 10 performance measures,  $J_i$ ,  $1 \leq i \leq 10$ , as summarized below.

#### RMS responses to Kanai–Tajimi excitation spectrum

The first five performance measures are root mean square responses to a family of Kanai–Tajimi excitation spectra.<sup>21</sup> The two-sided spectral density  $S_{\ddot{x}_g \ddot{x}_g}(\omega)$  is given by

$$S_{\ddot{x}_g \ddot{x}_g}(\omega) = S_0 \frac{4\zeta_g^2 \omega_g^2 \omega^2 + \omega_g^4}{(\omega^2 - \omega_g^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2}, \quad S_0 = \frac{s_0 \zeta_g}{\pi \omega_g (4\zeta_g^2 + 1)} \quad (2)$$

where the frequency  $\omega_g$  and damping ratio  $\zeta_g$  of the bedrock–ground connection are in given ranges ( $\zeta_g \in [0.3, 0.75]$ ,  $\omega_g$  as per Table I). (The intensity of the excitation is such that  $\sigma_{\ddot{x}_g}$ , the root mean square ground acceleration, is given in Table I and is independent of  $\omega_g$  and  $\zeta_g$ .) The performance measures, weighted by the non-dimensionalization parameters  $j_i$  in Table I, are root mean square structural response (maximum inter-story drifts and maximum floor acceleration) and actuator response (actuator displacement, velocity, and acceleration or force), defined by

$$J_1 = \max_{\omega_g, \zeta_g, i} \frac{\sigma_{d_i}}{j_1}, \quad J_2 = \max_{\omega_g, \zeta_g, i} \frac{\sigma_{\ddot{x}_m^a}}{j_2}, \quad J_3 = \max_{\omega_g, \zeta_g} \frac{\sigma_{x_m}}{j_3} \quad (3)$$

$$J_4 = \max_{\omega_g, \zeta_g} \frac{\sigma_{\dot{x}_m}}{j_4}, \quad J_5 = \max_{\omega_g, \zeta_g} \frac{\sigma_{\ddot{x}_m^a} \text{ or } \sigma_f}{j_5} \quad (4)$$

with the constraints

$$C_1 = \max_{\omega_g, \zeta_g} \frac{\sigma_u}{\sigma_u^{\text{MAX}}} \leq 1, \quad C_2 = \max_{\omega_g, \zeta_g} \left[ \frac{\sigma_{\ddot{x}_m^a}}{\sigma_{\ddot{x}_m^a}^{\text{MAX}}} \text{ or } \frac{\sigma_f}{\sigma_f^{\text{MAX}}} \right] \leq 1, \quad C_3 = \max_{\omega_g, \zeta_g} \frac{\sigma_{x_i}}{\sigma_{x_i}^{\text{MAX}}} \leq 1 \quad (5)$$

where  $\sigma_u^{\text{MAX}} = 1 \text{ V}$ ,  $\sigma_{\ddot{x}_m^a}^{\text{MAX}} = 2g$ ,  $\sigma_f^{\text{MAX}} = 4 \text{ kN}$ , and  $g = 9.8 \text{ m/s}^2$ . RMS responses are computed via a SIMULINK<sup>®</sup> block similar to that in Spencer *et al.*<sup>1,2</sup> up to 300 (AMD) or 750 (TEN) secs.

#### Peak responses to NS records of the 1940 El Centro and 1968 Hachinohe earthquakes

The remaining five performance measures are peak responses to the scaled ground acceleration records of two actual earthquakes. (The records and appropriate scaling were defined in the benchmark.<sup>20</sup>) The

performance measures are

$$J_6 = \max_{\substack{t,i \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|d_i(t)|}{j_6}, \quad J_7 = \max_{\substack{t,i \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|\ddot{x}_i^a(t)|}{j_7}, \quad J_8 = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|x_m(t)|}{j_8} \quad (6)$$

$$J_9 = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|\dot{x}_m(t)|}{j_9}, \quad J_{10} = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|\ddot{x}_m^a(t) \text{ or } f(t)|}{j_{10}} \quad (7)$$

with constraints

$$C_4 = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|u(t)|}{u^{\text{MAX}}} \leq 1, \quad C_5 = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \left[ \frac{|\ddot{x}_m^a(t)|}{\ddot{x}_m^{\text{aMAX}}} \text{ or } \frac{|f(t)|}{f^{\text{MAX}}} \right] \leq 1, \quad C_6 = \max_{\substack{t \\ \text{El Centro} \\ \text{Hachinohe}}} \frac{|x_m(t)|}{x_m^{\text{MAX}}} \leq 1 \quad (8)$$

where  $u^{\text{MAX}} = 3 \text{ V}$ ,  $\ddot{x}_m^{\text{aMAX}} = 6g$ , and  $f^{\text{MAX}} = 12 \text{ kN}$ .

#### *Performance of a 'sample' LQG controller for the active mass driver system*

One 'sample' controller, based on an LQG design that weights the three floor accelerations equally and feeds back the accelerations of the three floors and the active mass, was computed by Spencer *et al.*<sup>1</sup> the resulting values of the performance measures are given in Table II.

### NOTATION AND MATHEMATICAL DEFINITIONS

Various signal and system norms are used in formulating the multiobjective optimization problem to solve for mixed-norm optimal controllers. A brief mathematical description of these norms and their use in the current problem are as follows.

Let  $\mathbf{x}(t)$  be a real-valued vector process in  $\mathfrak{R}^m$ , with its discrete-time correspondent,  $\mathbf{x}(k)$ , with sampling time  $T$ . The  $p$ -norm of this signal may be defined as

$$\|\mathbf{x}\|_p = \left[ \int_{-\infty}^{\infty} \sum_{i=1}^m |x_i(t)|^p dt \right]^{1/p}, \quad \|\mathbf{x}\|_p = \left[ T \sum_{k=-\infty}^{\infty} \sum_{i=1}^m |\mathbf{x}_i(k)|^p \right]^{1/p} \quad (9)$$

The special cases for 1-, 2-, and  $\infty$ -norms are

$$\begin{aligned} \|\mathbf{x}\|_1 &= \int_{-\infty}^{\infty} \sum_{i=1}^m |x_i(t)| dt, & \|\mathbf{x}\|_1 &= T \sum_{k=-\infty}^{\infty} \sum_{i=1}^m |\mathbf{x}_i(k)| \\ \|\mathbf{x}\|_2^2 &= \int_{-\infty}^{\infty} \mathbf{x}^*(t) \mathbf{x}(t) dt, & \|\mathbf{x}\|_2^2 &= T \sum_{k=-\infty}^{\infty} \mathbf{x}^*(k) \mathbf{x}(k) \end{aligned} \quad (10)$$

$$\|\mathbf{x}\|_{\infty} = \sup_t \max_i |x_i(t)|, \quad \|\mathbf{x}\|_{\infty} = \sup_k \max_i |\mathbf{x}_i(k)|$$

where  $(\ )^*$  denotes complex-conjugate transpose. (The sampling time is included in the discrete-time signal norms so that, assuming  $T$  is sufficiently small, the resulting norms are equivalent to that of the continuous-time signal). A continuous-time signal is in  $L_p$  (a Lebesgue space), or a discrete-time signal in  $l_p$ , if its  $p$ -norm is finite.

Table I. Performance measure normalization parameters and benchmark constants

Actuator	$s_0$	$\omega_g$	$\sigma_{\ddot{x}_e}$	$j_1$	$j_2$	$j_3$	$j_4$	$j_5$
AMD	$0.03 \text{ V}^2$	$[20, 120] \frac{\text{rads}}{\text{sec}}$	$0.12g$	$1.31 \text{ cm}$	$1.79g$	$1.31 \text{ cm}$	$47.9 \text{ cm/sec}$	$1.79g$
TEN	$587.5 (\text{mV})^2$	$[8, 50] \frac{\text{rads}}{\text{sec}}$	$0.034g$	$2.34 \text{ cm}$	$0.485g$	$2.34 \text{ cm}$	$33.3 \text{ cm/sec}$	$289 \text{ kN}$
Actuator	Earthquake	$\sigma_{x_m}^{\text{MAX}}$	$x_m^{\text{MAX}}$	$j_6$	$j_7$	$j_8$	$j_9$	$j_{10}$
AMD	El Centro	3 cm	9 cm	$3.37 \text{ cm}$	$5.05g$	$3.37 \text{ cm}$	$131 \text{ cm/sec}$	$5.05g$
	Hachinohe			$1.66 \text{ cm}$	$2.58g$	$1.66 \text{ cm}$	$58.3 \text{ cm/sec}$	$2.58g$
TEN	El Centro	1 cm	3 cm	$6.45 \text{ cm}$	$1.57g$	$6.45 \text{ cm}$	$99.9 \text{ cm/sec}$	$289 \text{ kN}$
	Hachinohe			$3.78 \text{ cm}$	$0.778g$	$3.78 \text{ cm}$	$56.1 \text{ cm/sec}$	

Table II. Performance of sample LQG controller on the AMD system

$J_1$	$J_2$	$J_3$	$J_4$	$J_5$	$C_1$	$C_2$	$C_3$
0.283	0.440	0.510	0.513	0.628	0.143	0.56	0.224
$J_6$	$J_7$	$J_8$	$J_9$	$J_{10}$	$C_4$	$C_5$	$C_6$
0.456	0.711	0.670	0.775	1.34	0.175	0.605	0.222

Let  $\mathbf{G}(s)$  and  $\mathbf{G}(z)$  denote the continuous- and discrete-time transfer functions of a system, respectively, with corresponding impulse response functions,  $\hat{\mathbf{G}}(t)$  and  $\hat{\mathbf{G}}(k)$ . Furthermore, let  $\mathbf{u}$  and  $\mathbf{y}$  be the excitation and response (input and output of the system), respectively. Then several key norms maybe written as ( $\bar{\sigma}[\cdot]$  denotes the largest singular value)

$$\begin{aligned}
 L_1: \|\mathbf{G}\|_1 &= \max_i \int_{-\infty}^{\infty} \sum_{j=1}^n |\hat{g}_{ij}(t)| dt, & l_1: \|\mathbf{G}\|_1 &= \max_i T \sum_{k=-\infty}^{\infty} \sum_{j=1}^n |\hat{g}_{ij}(k)| \\
 H_2: \|\mathbf{G}\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} [\mathbf{G}^*(j\omega) \mathbf{G}(j\omega)] d\omega, & H_2: \|\mathbf{G}\|_2^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace} [\mathbf{G}^*(e^{j\omega}) \mathbf{G}(e^{j\omega})] d\omega \\
 H_{\infty}: \|\mathbf{G}\|_{\infty} &= \sup_{\omega} \bar{\sigma}[\mathbf{G}(j\omega)], & H_{\infty}: \|\mathbf{G}\|_{\infty} &= \sup_{\omega} \bar{\sigma}[\mathbf{G}(e^{j\omega})]
 \end{aligned} \tag{11}$$

The  $H_2$  norm of a system reflects the root mean square response to Gaussian white noise excitation, and thus it is central to control design for the benchmark problem; minimizing the closed-loop  $H_2$  norm  $\|\mathbf{G}\|_2$  will minimize the root mean square output  $\|\mathbf{y}\|_2$ . The  $H_{\infty}$  norm is a worst-case magnification of excitation 'energy' into the system response (i.e.  $\|\mathbf{G}\|_{\infty} = \max_{\mathbf{u} \in l_2} \|\mathbf{y}\|_2 / \|\mathbf{u}\|_2$ ); it also can be used to measure the worst-case sensitivity of the stability of the closed-loop system to unmodelled dynamics, such as high-frequency modes that are often neglected or have low signal-to-noise ratios (making them difficult to model accurately). Thus, to minimize the  $H_{\infty}$  norm around an unknown block of unmodelled dynamics for robust

control design will minimize the sensitivity of the closed-loop stability to the uncertainties in the model. Finally, the  $L_1$  norm (or the  $l_1$  norm for discrete-time systems) is the worst-case magnification of input magnitude; minimizing the  $L_1$  norm can often decrease peak responses.

### MULTIOBJECTIVE OPTIMAL CONTROL

The Mixed-Norm Toolbox<sup>18,19</sup>, also known as MXTTOOLS, is designed to solve mixed-norm, fixed-order, optimal control problems by choosing a controller of a selected order to minimize an  $H_2$  norm subject to one or more  $H_2$ ,  $L_1$  or  $l_1$ , and  $H_\infty$  norm constraints. Assuming a base  $H_2$  problem, one  $L_1$  or  $l_1$  constraint, and one  $H_\infty$  constraint, the problem is described by the interrelated systems in Figure 2 and the corresponding continuous-time state-space descriptions

$$\begin{aligned}\dot{\mathbf{x}}_2 &= \mathbf{A}_2 \mathbf{x}_2 + \mathbf{B}_w \mathbf{w} + \mathbf{B}_2 \mathbf{u}, & \dot{\mathbf{x}}_1 &= \mathbf{A}_1 \mathbf{x}_1 + \mathbf{B}_r \mathbf{r} + \mathbf{B}_1 \mathbf{u}, & \dot{\mathbf{x}}_\infty &= \mathbf{A}_\infty \mathbf{x}_\infty + \mathbf{B}_d \mathbf{d} + \mathbf{B}_\infty \mathbf{u} \\ \bar{\mathbf{z}} &= \mathbf{C}_z \mathbf{x}_2 + \mathbf{D}_{zw} \mathbf{w} + \mathbf{D}_{zu} \mathbf{u}, & \mathbf{m} &= \mathbf{C}_m \mathbf{x}_1 + \mathbf{D}_{mr} \mathbf{r} + \mathbf{D}_{mu} \mathbf{u}, & \mathbf{e} &= \mathbf{C}_e \mathbf{x}_\infty + \mathbf{D}_{ed} \mathbf{d} + \mathbf{D}_{eu} \mathbf{u} \\ \mathbf{y}_2 &= \mathbf{C}_2 \mathbf{x}_2 + \mathbf{D}_{yw} \mathbf{w} + \mathbf{D}_{yu} \mathbf{u}, & \mathbf{y}_1 &= \mathbf{C}_1 \mathbf{x}_1 + \mathbf{D}_{yr} \mathbf{r} + \mathbf{D}_{yu} \mathbf{u}, & \mathbf{y}_\infty &= \mathbf{C}_\infty \mathbf{x}_\infty + \mathbf{D}_{yd} \mathbf{d} + \mathbf{D}_{yu} \mathbf{u}\end{aligned}\quad (12)$$

(or similar discrete-time equations), where state vectors  $\mathbf{x}_2$ ,  $\mathbf{x}_1$ , and  $\mathbf{x}_\infty$  may have some or all states in common and the measurements  $\mathbf{y}_2$ ,  $\mathbf{y}_1$ , and  $\mathbf{y}_\infty$  are equal in the absence of disturbances  $\mathbf{w}$ ,  $\mathbf{r}$ , and  $\mathbf{d}$ , i.e.  $\mathbf{C}_2 \mathbf{x}_2 = \mathbf{C}_1 \mathbf{x}_1 = \mathbf{C}_\infty \mathbf{x}_\infty$ . The input  $\mathbf{w}$  is a zero-mean Gaussian white noise of unit intensity (i.e.  $E[\mathbf{w}(t)\mathbf{w}^T(s)] = \mathbf{I}\delta(t-s)$  or  $E[\mathbf{w}(k)\mathbf{w}^T(l)] = (1/T)\mathbf{I}\delta_{kl}$ , where  $T$  is the sampling period), the input  $\mathbf{r}$  is assumed to be an unknown magnitude-bounded signal with  $\|\mathbf{r}\|_\infty \leq 1$ , and the input  $\mathbf{d}$  is assumed to be an unknown energy-bounded signal with  $\|\mathbf{d}\|_2 \leq 1$ . No relationships are assumed between  $\mathbf{w}$ ,  $\mathbf{r}$ , and  $\mathbf{d}$  or between  $\bar{\mathbf{z}}$ ,  $\mathbf{m}$ , and  $\mathbf{e}$ .

The  $H_2/L_1/H_\infty$  (or  $H_2/l_1/H_\infty$ ) optimization problem is then to find an admissible, stabilizing controller  $\mathbf{K}$ , with state-space description  $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ , of a fixed order, that achieves

$$\inf_{\mathbf{K} \text{ admissible}} \|\mathbf{T}_{zw}\|_2 \quad \text{subject to} \quad \|\mathbf{T}_{mr}\|_1 \leq \nu \text{ and } \|\mathbf{T}_{ed}\|_\infty \leq \gamma \quad (13)$$

The requirements for solution with the fixed order at least that of the base  $H_2$  subproblem are

- (1) the base  $H_2$  subproblem must have  $(\mathbf{A}_2, \mathbf{B}_2)$  stabilizable and  $(\mathbf{A}_2, \mathbf{C}_2)$  detectable,
- (2) the base  $H_2$  subproblem must be well posed, i.e.,  $(\mathbf{I} + \mathbf{D}_c \mathbf{D}_{yu})$  is invertible,
- (3) the overall problem must be non-singular, and
- (4) for continuous-time systems,  $\mathbf{D}_{zw} + \mathbf{D}_{zu} \mathbf{D}_c \mathbf{D}_{yw} = 0$  so that  $\|\mathbf{T}_{zw}\|_2$  is finite.

For a controller of lesser order, an additional requirement is the existence of a stabilizing controller of the given order that satisfies the constraints.

MXTTOOLS parameterizes the controller in modal form to reduce the number of design variables (generally disallows repeated controller eigenvalues; one could modify MXTTOOLS to use a full state space, or any other, parametrization if repeated eigenvalues are thought to be required). The objective function, the constraints, and their gradients, are constructed<sup>17</sup> and passed to any constrained optimization algorithm. The default solver uses a sequential quadratic programming (SQP) method similar in function to MATLAB's Optimization Toolbox<sup>22</sup> `constr` function, but handles larger problems more effectively (`constr` often has difficulty maintaining a positive definite approximation to the Lagrangian Hessian for problems with more than 40 or 50 design variables).

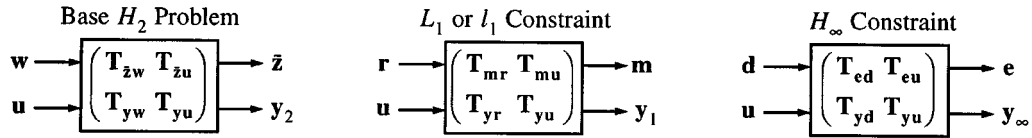


Figure 2. MxTOOLS plant with a base  $H_2$  system, an  $L_1$  or  $l_1$  constraint, and an  $H_\infty$  constraint

The formulation above assumes one  $L_1$  or  $l_1$  constraint and one  $H_\infty$  constraint. Similar formulations for an arbitrary number of constraints may be performed. The current implementation of MxTOOLS is limited to one MIMO  $L_1$  or  $l_1$  constraint, though this is of no difficulty since several  $L_1$  and  $l_1$  constraints can be combined into one with appropriate scaling.

A few caveats in the use of MxTOOLS must be noted. First, no information is available as to how distant a computed fixed-order solution is from the true optimal, order-free solution.<sup>19</sup> For the benchmark problem, this is not of great concern since the controller order is restricted. Second, the search for reduced-order controllers may be more susceptible to the problem of local minima, but may be significantly easier to solve due to the reduced search space dimension.<sup>19</sup> Third, the  $l_1$  norm tends to be quite conservative for most magnitude-bounded, but otherwise unknown, inputs, and therefore its values ought to be regarded in a relative, qualitative sense, rather than a measure of the actual performance on magnitude-bounded output constraints.<sup>17</sup>

The MxTOOLS software was a useful tool in solving the multiobjective problems in this study, though a fair amount of ‘tweaking’ and a few bug fixes were necessary in the process.

## FORMULATION OF THE SUBPROBLEMS

The benchmark problem can be augmented by various static and dynamic weights to transform it into a form suitable for a mixed-norm optimal controller search. One aspect of the benchmark problem rather indirectly addressed in the control design here is the 200  $\mu$ s computational delay at the plant input (time to compute the control action and perform the A/D and D/A conversions); it is included here only in the  $H_\infty$  subproblem as a part of the plant uncertainty. If the delay were larger or considered more important, the base plant could be explicitly augmented from the start (at the plant input) with a Padé approximation of the delay.<sup>23</sup>

Some of the formulation given below is in continuous-time terms, but all such transfer functions were converted to discrete-time for the control optimization problem.

### Base $H_2$ problem

The base  $H_2$  problem can be formulated by adding to the base plant various weighting functions for normalization, as shown in Figure 3, where  $\mathbf{w}$  is a  $7 \times 1$  vector of zero-mean, Gaussian white noises with  $E[\mathbf{w}(t)\mathbf{w}^T(t + \tau)] = \mathbf{I}\delta(\tau)$ . One element, or channel, of  $\mathbf{w}$  is filtered through the second-order system  $W_{KT}(s)$  to model the Kanai–Tajimi spectrum

$$W_{KT}(s, \omega_g, \zeta_g) = \sqrt{2\pi S_0} \frac{2\zeta_g \omega_g s + \omega_g^2}{s^2 + 2\zeta_g \omega_g s + \omega_g^2} \quad (14)$$

using the ground parameters  $\omega_g$  and  $\zeta_g$  that cause the worst uncontrolled response ( $\zeta_g^{\text{worst}} = 0.3$ ;  $\omega_g^{\text{worst}} = 37.28$  rad/sec for the active mass driver system and  $\omega_g^{\text{worst}} = 14.5$  rad/sec for the active tendon

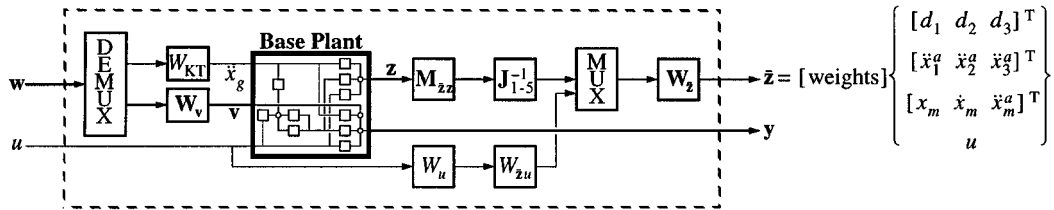


Figure 3. Block diagram of the base  $H_2$  problem. (Note: "DEMUX" is a demultiplexer that separates two sets of signals in a multi-signal line; "MUX", or multiplexer, combines several channels into one vector signal.)

system). The remaining elements of  $\mathbf{w}$  are weighted to model the sensor noise. At the output side,  $W_{zu}$  is a weight on the control effort and  $W_u$  is a normalization factor.  $\mathbf{M}_{zz}$  is a matrix used to rearrange the outputs of the base plant into the elements to be regulated;  $\mathbf{Z}_V^{-1}$  converts from Volts to the dimensional units and then  $\mathbf{J}_{1-5}^{-1}$  non-dimensionalizes.  $\mathbf{W}_z$  is a diagonal matrix to weight the normalized criteria differently as required for improved controller performance (the identity matrix was found acceptable in the final designs herein). The resulting output vector  $\bar{\mathbf{z}}$  contains weighted forms of the three inter-story drifts, the three floor accelerations, and the actuator relative displacement, relative velocity, absolute acceleration (actuator force for the active tendon system), and control effort.

$$\mathbf{J}_{1-5} = \text{diag} \begin{Bmatrix} [j_1 & j_1 & j_1]^T \\ [j_2 & j_2 & j_2]^T \\ [j_3 & j_4 & j_5]^T \end{Bmatrix}, \quad \mathbf{W}_v = 0.01 \sqrt{T} \mathbf{I}, \quad W_u = 1/\sigma_u^{\text{MAX}}$$

$$\mathbf{Z}_V^{\text{AMD}} = \text{diag} \begin{Bmatrix} 0.51 \text{ V/cm} & [1 & 1 & 1 & 1]^T \\ 0.0204 \text{ Vs/cm} & [1 & 1 & 1 & 1]^T \\ \text{V/g} & [1 & 1 & 1 & 0.1]^T \end{Bmatrix}, \quad \mathbf{M}_{zz} = \begin{bmatrix} 1 & 0 & 0 & 0 & & & & \\ -1 & 1 & 0 & 0 & \mathbf{0} & & & \mathbf{0} \\ 0 & -1 & 1 & 0 & & & & \\ & & & & & & 1 & 0 & 0 & 0 \\ & \mathbf{0} & & \mathbf{0} & & & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{Z}_V^{-1}$$

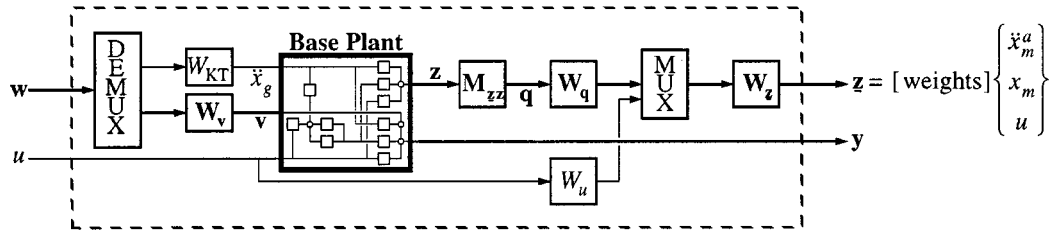
$$\mathbf{Z}_V^{\text{TEN}} = \text{diag} \begin{Bmatrix} 0.25 \text{ V/cm} & [1 & 1 & 1 & 31.48]^T \\ 0.0102 \text{ Vs/cm} & [1 & 1 & 1 & 38.627]^T \\ 0.5 \text{ V/g} & [1 & 1 & 1 & 0.444 \text{ g/kN}]^T \end{Bmatrix}, \quad \mathbf{q} = \begin{Bmatrix} \ddot{x}_m^a \text{ or } f \\ x_m \end{Bmatrix}, \quad \mathbf{M}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{Z}_V^{-1}$$

### $H_2$ constraints

The benchmark requires that the root mean square controller output, actuator displacement, and actuator acceleration/force be less than critical values, to remain within the operational range of the actuator. This can be formulated as an  $H_2$  constraint on the system shown in Figure 4, where the new weights are given by

$$\mathbf{W}_q = \begin{bmatrix} 1/\sigma_{\ddot{x}_m}^{\text{MAX}} \text{ or } 1/\sigma_f^{\text{MAX}} & 0 \\ 0 & 1/\sigma_{x_m}^{\text{MAX}} \end{bmatrix}, \quad \mathbf{q} = \begin{Bmatrix} \ddot{x}_m^a \text{ or } f \\ x_m \end{Bmatrix}, \quad \mathbf{M}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{Z}_V^{-1}$$



Figure 4. Block diagram of the  $H_2$  constraints

and where  $\mathbf{W}_z$  is a diagonal matrix to be determined for best performance. More properly, three  $H_2$  constraints, one for each element of  $\mathbf{z}$  will be used herein (though one may substitute the single constraint in the interests of computational feasibility).

### $l_1$ constraints

The use of  $l_1$  constraints permits the inclusion of the hard peak constraints of the actuator (to avoid saturation) as well as to facilitate bounding the performance of the peak response portion of the benchmark as measured via  $J_6$ – $J_{10}$ . The block diagram of the  $l_1$  constraint subproblem is shown in Figure 5, where the weights, chosen to reflect actuator peak constraints and the desired performance of the peak responses, are given by

$$W_{mu} = \max_{\substack{\text{El Centro} \\ \text{Hachinohe}}} \left[ \frac{\|\ddot{x}_g\|_{\infty}}{u^{\text{MAX}}} \right], \quad \mathbf{J}_{6-10} = \min_{\substack{\text{El Centro} \\ \text{Hachinohe}}} \frac{1}{\|\ddot{x}_g\|_{\infty}} \text{diag} \left\{ \begin{array}{c} [j_6 \ j_6 \ j_6]^T \\ [j_7 \ j_7 \ j_7]^T \\ \max(j_8, x_m^{\text{MAX}}) \\ j_9 \\ \max(j_{10}, \ddot{x}_m^{\text{MAX}} \text{ or } f^{\text{MAX}}) \end{array} \right\} \quad (17)$$

$$\mathbf{M}_{mz} = \begin{bmatrix} \mathbf{M}_{zz} \\ \mathbf{M}_{zz} \end{bmatrix},$$

The output vector  $\bar{\mathbf{m}}$ , the system responses whose peak magnitudes should be considered or minimized, is the same as  $\bar{\mathbf{z}}$  in the base  $H_2$  problem but with different weights. The input  $r$  is the disturbance, which should be a bounded-magnitude signal for this subproblem (i.e. in  $l_{\infty}$ ); the problem was studied with two input weights  $W_r$ : unity and a Kanai–Tajimi filter to more closely model earthquake input spectra (in which case  $r$  corresponds to the acceleration of bedrock in the Kanai–Tajimi earthquake model). The matrix  $\mathbf{W}_m$  serves as a final scaling matrix and to select elements to include in the output vector  $\mathbf{m}$ .

It is observed in evaluation of several controllers that four of the ten elements of  $\bar{\mathbf{m}}$  were critical (weighted measures of the 2nd and 3rd floor accelerations,  $\bar{m}_5$  and  $\bar{m}_6$ , actuator velocity  $\bar{m}_8$ , and actuator acceleration/force  $\bar{m}_{10}$ ); due to the expense of computing the  $l_1$  norm and its gradient, the matrix  $\mathbf{W}_m$  was used to select only those four outputs in  $\mathbf{m}$  during the optimization.

### $H_{\infty}$ constraints

One typical purpose for involving  $H_{\infty}$  in many problems is to minimize the error in tracking given command inputs, generally by minimizing the weighted sensitivity of the system. The problem at hand,



## CONTROLLER DESIGNS: ACTIVE MASS DRIVER SYSTEM

*Performance with zero controller*

In order to properly judge superior control designs, the performance of the system with the 'zero controller' (a 0V command signal always sent to the actuator) was first evaluated. Since the 'closed-loop' system is linear, the RMS responses to the Kanai–Tajimi spectra can be efficiently computed by solving a Lyapunov equation (if a Multi-Input–Single-Output (MISO) system has Laplace transform  $Y(s) = \mathbf{G}(s)\mathbf{W}(s)$ , where  $\mathbf{w}(t)$  is a zero-mean, Gaussian white noise vector process with  $E[\mathbf{w}(t)\mathbf{w}^T(s)] = \mathbf{I}\delta(t-s)$ , then the expected output is  $E[y^2(t)] = y_{\text{RMS}}^2 = \|\mathbf{G}\|_2^2$ ). The peak responses to the known earthquakes can be simulated quickly via `lsim` in MATLAB®.

The resulting performance data are given in column #1 of Table IV (including the Kanai–Tajimi parameters  $\omega_g$  and  $\zeta_g$  that maximized the RMS values). For purposes of verification, the worst-case RMS and peak third-floor responses, used in the  $j_i$  performance weights, were computed and are shown in Table III; they are fairly consistent with the  $j_i$  given above in Table I.

*Performance with sample controller*

In order to verify that the evaluation software was performing properly, the sample controller supplied with the benchmark proposal was evaluated. The results, shown in column #2 of Table IV, are consistent with those detailed in the benchmark and given above in Table II (with the exception of  $C_5$ , which in Table II may reflect just one of the two earthquakes in the benchmark), demonstrating that the evaluation software operates satisfactorily.

*Base  $H_2$  (LQG) design*

The primary parameter to be determined in a base  $H_2$  design is the relative weight on the control effort,  $W_{zu}$ . Full-order, discrete-time  $H_2$ -optimal controllers were computed using a wide range of control effort values to minimize  $\|\mathbf{T}_{zw}\|_2$ . The resulting RMS performance and constraint values, assuming the system is sufficiently linear to apply the appropriate Lyapunov equations, are shown in Figure 7. It is a worthwhile tradeoff to allow a larger control effort and greater actuator dynamics for the sake of reducing the larger performance and constraint values by letting  $W_{zu}$  be unity. A choice of  $W_{zu} < 1$  gains little in the way of the RMS performance and constraints, but may limit the ability to increase robustness and output magnitudes in subsequent optimization.

Two initial reduced order  $H_2$  controllers were computed, one by a Hankel norm model reduction of the full  $H_2$  controller, and the other by first reducing the plant and then designing an  $H_2$  controller. These two designs were then used as initial conditions in a minimization of the  $H_2$  norm of the closed-loop formed with the full-order plant and the reduced-order controller. The performance on the benchmark for the resulting (reduced-order)  $H_2$  optimal controller is given in column #3 of Table IV.

Two additional reduced-order controllers are used as starting points in the  $H_2/H_\infty$  optimization below, computed in a manner similar, but doing  $H_\infty$  control designs by minimizing  $\|\mathbf{T}_{ed}\|_\infty$ .

The base  $H_2$  problem contains somewhat conflicting goals, since the response of the structure and the response of the actuator cannot be simultaneously minimized. Figure 8 shows the tradeoff between the RMS structural response  $\|\mathbf{T}_{zw}^{\text{Str}}\|_2$  (simply the first six outputs of  $\bar{\mathbf{z}}$  in the base  $H_2$  plant) and the RMS actuator response  $\|\mathbf{T}_{zw}^{\text{Act}}\|_2$  (the three actuator responses in  $\bar{\mathbf{z}}$ ). (The constraints were ignored in the minimization, but

Table III. Worst-case third-story responses with zero controller

Actuator	$\sigma_{x_3}^{\text{worst}}$	$\sigma_{\dot{x}_3}^{\text{worst}}$	$\sigma_{\ddot{x}_3}^{\text{worst}}$	Earthquake
AMD	1.312911 cm	47.8787 cm/sec	1.787342g	Kanai-Tajimi
	$\omega_g = 37.2674$ rad/sec $\zeta_g = 0.30000$	$\omega_g = 37.3078$ rad/sec $\zeta_g = 0.30000$	$\omega_g = 37.3000$ rad/sec $\zeta_g = 0.30000$	
TEN	2.386049 cm	33.2794 cm/sec	0.485246g	Kanai-Tajimi
	$\omega_g = 14.5478$ rad/sec $\zeta_g = 0.30000$	$\omega_g = 14.5764$ rad/sec $\zeta_g = 0.30000$	$\omega_g = 14.5635$ rad/sec $\zeta_g = 0.30000$	
Actuator	$x_3^{\text{worst}}$	$\dot{x}_3^{\text{worst}}$	$\ddot{x}_3^{\text{worst}}$	Earthquake
AMD	3.444070 cm	131.3235 cm/sec	5.051008g	El Centro
	1.662584 cm	58.6767 cm/sec	2.696785g	Hachinohe
TEN	6.574704 cm	99.9140 cm/sec	1.573521g	El Centro
	3.850504 cm	56.0717 cm/sec	0.777907g	Hachinohe

are noted in the graph where violations occurred.) The benchmark performance of the controller that minimized the RMS actuator response is given in column #4 of Table IV, while column #5 gives the performance of the controller that minimized the RMS structural response while not violating RMS constraints (it did violate the peak active mass acceleration, but it is useful to see what the performance would be without the peak constraints). If the relative importance of these two sets of responses were known, one could appropriately weight the structural and actuator responses in the base  $H_2$  problem; in the mixed-norm designs below, however, they are weighted equally.

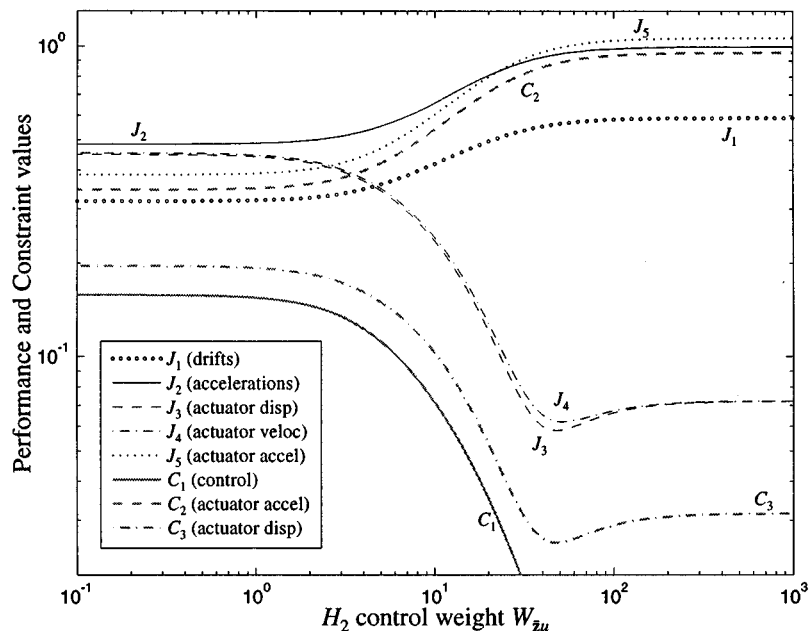
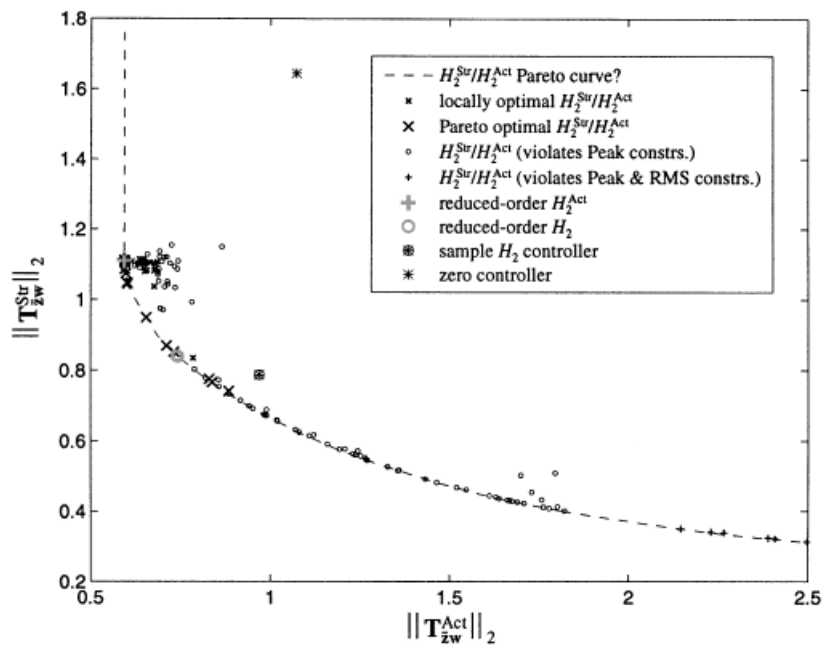
### Mixed $H_2/H_\infty$ design

About 100 different control designs were computed via the optimization software to study the tradeoffs between  $H_2$  and  $H_\infty$  performance. The norms of the resulting closed-loop systems are plotted in Figure 9; some of the designs obviously stalled at local minima, but a number of them demonstrate quite clearly that significant robust stability and robust performance can be obtained with near  $H_2$  performance. (Note that the Pareto curve is approximate since the points from computed control designs are upper and right bounds on the curve, while constant lines at the closed-loop  $\infty$ -norm with the full-order  $H_\infty$  controller and the 2-norm with the  $H_2$  controller are left and lower bounds, respectively.) The two circled larger  $\times$ 's near the minimum  $H_2$  have 2-norms only 0.86 and 0.015 per cent above minimum, but at one thirteenth and one-eighth of the infinity norm of the reduced-order  $H_2$  optimal design. Such design points introduce significant robustness with negligible changes in RMS outputs to white noise excitation. The third circled large  $\times$  has the opposite tradeoff, with a closed-loop infinity norm nearly that of the  $H_\infty$  optimal controller, but with a RMS output over 30 per cent less.

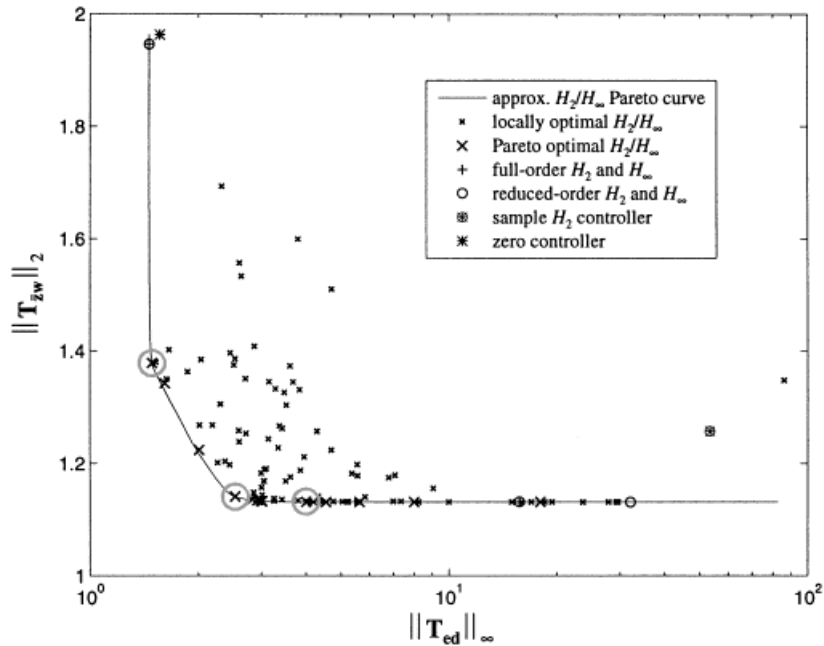
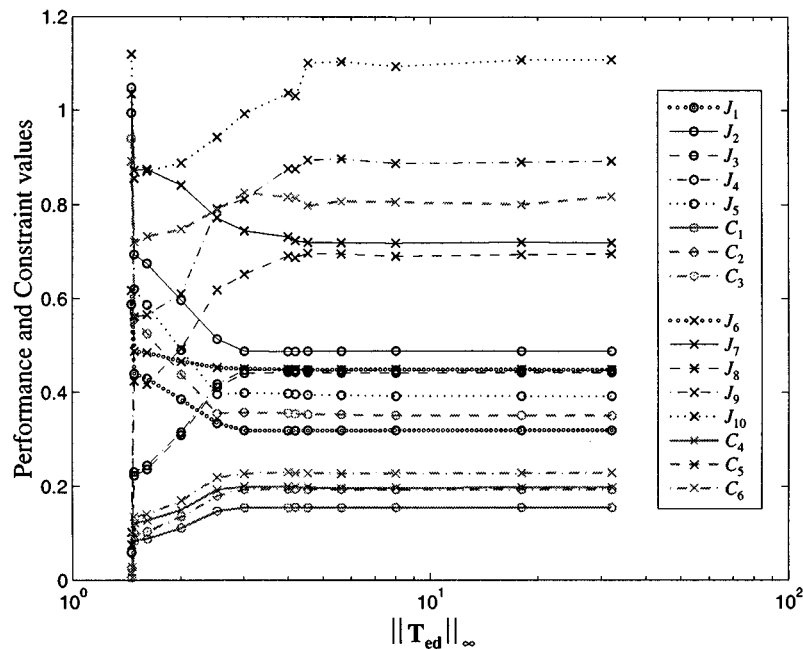
The benchmark performance values for several of the (near-) Pareto optimal designs are graphed in Figure 10, where it can be seen that the performance changes little for vast robustness improvements. The performance on the benchmark problem for the first of the above-mentioned designs is given in column #6 of Table IV, and its input loop gain is shown in Figure 11; this design far exceeds the robustness requirements in the benchmark description with little or no performance degradation. The performance of the

Table IV. Performance and constraints of the AMD and TEN systems with various controllers  
 The worst-case ( $\omega_g, \zeta_g$ ) locations were found for each RMS value, using 300 (AMD) or 750 (TEN) seconds of response to a Kanai–Tajimi excitation, and are given below the corresponding  $J$  and  $C$  values ( $\omega_g$  is in rad/sec)

Sys Name Note:	1 AMD Zero	2 AMD Sample $H_2$ min. accel. responses	3 AMD Base $H_2$ min. struct. & actuator responses	4 AMD $H_2^{\text{st}}$ min. actuator responses	5 AMD $H_2^{\text{str}}$ min. structural responses	6 AMD $H_2/H_\infty$ #1 stability/ perform. robustness	7 AMD $H_2/H_\infty$ #2 least mean sq. perf. & cnstr. vals	8 AMD $H_2/H_\infty^{\text{rs}}$ stability robustness	9 TEN Zero	10 TEN Base $H_2$ min. struct. & actuator responses	11 TEN $H_2/H_\infty$ stability/ perform. robustness
$J_1$	<b>0.589235</b> $\omega_g = 37.2604$ $\zeta_g = 0.30000$	<b>0.284008</b> $\omega_g = 37.1276$ $\zeta_g = 0.30000$	<b>0.331097</b> $\omega_g = 37.2728$ $\zeta_g = 0.30000$	<b>0.398656</b> $\omega_g = 37.3087$ $\zeta_g = 0.30000$	<b>0.148870</b> $\omega_g = 36.8128$ $\zeta_g = 0.30000$	<b>0.327632</b> $\omega_g = 37.2682$ $\zeta_g = 0.30000$	<b>0.324451</b> $\omega_g = 37.2665$ $\zeta_g = 0.30000$	<b>0.313034</b> $\omega_g = 37.2563$ $\zeta_g = 0.30000$	<b>0.469652</b> $\omega_g = 14.5492$ $\zeta_g = 0.30000$	<b>0.167053</b> $\omega_g = 14.4392$ $\zeta_g = 0.30000$	<b>0.167236</b> $\omega_g = 14.4387$ $\zeta_g = 0.30000$
$J_2$	<b>0.998515</b> $\omega_g = 37.3000$ $\zeta_g = 0.30000$	<b>0.439718</b> $\omega_g = 37.0922$ $\zeta_g = 0.30000$	<b>0.476400</b> $\omega_g = 37.2648$ $\zeta_g = 0.30000$	<b>0.632393</b> $\omega_g = 37.4397$ $\zeta_g = 0.30000$	<b>0.221632</b> $\omega_g = 36.5708$ $\zeta_g = 0.30000$	<b>0.503912</b> $\omega_g = 37.2700$ $\zeta_g = 0.30000$	<b>0.498581</b> $\omega_g = 37.2628$ $\zeta_g = 0.30000$	<b>0.479473</b> $\omega_g = 37.2476$ $\zeta_g = 0.30000$	<b>1.000507</b> $\omega_g = 14.5635$ $\zeta_g = 0.30000$	<b>0.359133</b> $\omega_g = 14.4931$ $\zeta_g = 0.30000$	<b>0.359519</b> $\omega_g = 14.4929$ $\zeta_g = 0.30000$
$J_3$	<b>0.072189</b> $\omega_g = 37.2522$ $\zeta_g = 0.30000$	<b>0.511574</b> $\omega_g = 37.0152$ $\zeta_g = 0.30000$	<b>0.435891</b> $\omega_g = 37.2366$ $\zeta_g = 0.30000$	<b>0.283548</b> $\omega_g = 37.3355$ $\zeta_g = 0.30000$	<b>1.072474</b> $\omega_g = 36.3575$ $\zeta_g = 0.30000$	<b>0.404407</b> $\omega_g = 37.3585$ $\zeta_g = 0.30000$	<b>0.398811</b> $\omega_g = 37.2862$ $\zeta_g = 0.30000$	<b>0.433318</b> $\omega_g = 37.2743$ $\zeta_g = 0.30000$	<b>0.008641</b> $\omega_g = 14.5400$ $\zeta_g = 0.30000$	<b>0.025988</b> $\omega_g = 14.7564$ $\zeta_g = 0.30000$	<b>0.025956</b> $\omega_g = 14.7542$ $\zeta_g = 0.30000$
$J_4$	<b>0.082320</b> $\omega_g = 37.4052$ $\zeta_g = 0.30000$	<b>0.512567</b> $\omega_g = 37.1222$ $\zeta_g = 0.30000$	<b>0.440530</b> $\omega_g = 37.4334$ $\zeta_g = 0.30000$	<b>0.288202</b> $\omega_g = 37.4246$ $\zeta_g = 0.30000$	<b>1.047233</b> $\omega_g = 37.4442$ $\zeta_g = 0.30000$	<b>0.410838</b> $\omega_g = 37.5295$ $\zeta_g = 0.30000$	<b>0.403989</b> $\omega_g = 37.3699$ $\zeta_g = 0.30000$	<b>0.438351</b> $\omega_g = 37.4553$ $\zeta_g = 0.30000$	<b>0.008651</b> $\omega_g = 14.5763$ $\zeta_g = 0.30000$	<b>0.065933</b> $\omega_g = 50.0000$ $\zeta_g = 0.75000$	<b>0.061770</b> $\omega_g = 50.0000$ $\zeta_g = 0.75000$
$J_5$	<b>1.067546</b> $\omega_g = 37.3060$ $\zeta_g = 0.30000$	<b>0.626813</b> $\omega_g = 37.0076$ $\zeta_g = 0.30000$	<b>0.368803</b> $\omega_g = 37.9657$ $\zeta_g = 0.30000$	<b>0.389057</b> $\omega_g = 37.2734$ $\zeta_g = 0.30000$	<b>1.056882</b> $\omega_g = 38.6359$ $\zeta_g = 0.30000$	<b>0.366769</b> $\omega_g = 37.8947$ $\zeta_g = 0.30000$	<b>0.402087</b> $\omega_g = 37.2931$ $\zeta_g = 0.30000$	<b>0.370105</b> $\omega_g = 38.1447$ $\zeta_g = 0.30000$	<b>0.027074</b> $\omega_g = 14.5386$ $\zeta_g = 0.30000$	<b>0.010131</b> $\omega_g = 14.2808$ $\zeta_g = 0.30000$	<b>0.010132</b> $\omega_g = 14.2813$ $\zeta_g = 0.30000$
$J_6$	<b>0.620058</b>	<b>0.455587</b>	<b>0.449376</b>	<b>0.912614</b>	<b>0.356579</b>	<b>0.453735</b>	<b>0.469161</b>	<b>0.448668</b>	<b>0.478946</b>	<b>0.294108</b>	<b>0.298966</b>
$J_7$	<b>0.717755</b>	<b>0.710194</b>	<b>0.715053</b>	<b>0.710194</b>	<b>0.559262</b>	<b>0.770059</b>	<b>0.805363</b>	<b>0.749228</b>	<b>0.822220</b>	<b>0.775928</b>	<b>0.771151</b>
$J_8$	<b>0.077206</b>	<b>0.667990</b>	<b>0.689468</b>	<b>0.397785</b>	<b>3.159213</b>	<b>0.621133</b>	<b>0.547477</b>	<b>0.687785</b>	<b>0.009316</b>	<b>0.057316</b>	<b>0.057130</b>
$J_9$	<b>0.083293</b>	<b>0.775300</b>	<b>0.880412</b>	<b>0.445912</b>	<b>3.264125</b>	<b>0.783109</b>	<b>0.586780</b>	<b>0.954007</b>	<b>0.010837</b>	<b>0.203174</b>	<b>0.184940</b>
$J_{10}$	<b>1.142254</b>	<b>1.336047</b>	<b>1.031068</b>	<b>0.513109</b>	<b>4.533990</b>	<b>0.858705</b>	<b>0.682718</b>	<b>1.156828</b>	<b>0.079961</b>	<b>0.040999</b>	<b>0.039997</b>
$C_1$	<b>0</b> $\omega_g = 37.1436$ $\zeta_g = 0.30000$	<b>0.143045</b> $\omega_g = 37.1436$ $\zeta_g = 0.30000$	<b>0.147656</b> $\omega_g = 37.5456$ $\zeta_g = 0.30000$	<b>0.110122</b> $\omega_g = 37.6263$ $\zeta_g = 0.30000$	<b>0.365361</b> $\omega_g = 36.7604$ $\zeta_g = 0.30000$	<b>0.140550</b> $\omega_g = 37.6811$ $\zeta_g = 0.30000$	<b>0.127599</b> $\omega_g = 37.5930$ $\zeta_g = 0.30000$	<b>0.146899</b> $\omega_g = 37.5627$ $\zeta_g = 0.30000$	<b>0</b> $\omega_g = 37.5930$ $\zeta_g = 0.30000$	<b>0.490599</b> $\omega_g = 14.6978$ $\zeta_g = 0.30000$	<b>0.490159</b> $\omega_g = 14.6972$ $\zeta_g = 0.30000$
$C_2$	<b>0.955454</b> $\omega_g = 37.3060$ $\zeta_g = 0.30000$	<b>0.560998</b> $\omega_g = 37.0076$ $\zeta_g = 0.30000$	<b>0.330079</b> $\omega_g = 37.9657$ $\zeta_g = 0.30000$	<b>0.348206</b> $\omega_g = 37.2734$ $\zeta_g = 0.30000$	<b>0.945909</b> $\omega_g = 38.6359$ $\zeta_g = 0.30000$	<b>0.328258</b> $\omega_g = 37.8947$ $\zeta_g = 0.30000$	<b>0.359868</b> $\omega_g = 37.2931$ $\zeta_g = 0.30000$	<b>0.331244</b> $\omega_g = 38.1447$ $\zeta_g = 0.30000$	<b>1.956081</b> $\omega_g = 14.5386$ $\zeta_g = 0.30000$	<b>0.732025</b> $\omega_g = 14.2808$ $\zeta_g = 0.30000$	<b>0.732070</b> $\omega_g = 14.2813$ $\zeta_g = 0.30000$
$C_3$	<b>0.031522</b> $\omega_g = 37.2522$ $\zeta_g = 0.30000$	<b>0.223387</b> $\omega_g = 37.0152$ $\zeta_g = 0.30000$	<b>0.190339</b> $\omega_g = 37.2366$ $\zeta_g = 0.30000$	<b>0.123816</b> $\omega_g = 37.3355$ $\zeta_g = 0.30000$	<b>0.468313</b> $\omega_g = 36.3575$ $\zeta_g = 0.30000$	<b>0.176591</b> $\omega_g = 37.3585$ $\zeta_g = 0.30000$	<b>0.174147</b> $\omega_g = 37.2862$ $\zeta_g = 0.30000$	<b>0.189215</b> $\omega_g = 37.2743$ $\zeta_g = 0.30000$	<b>0.020219</b> $\omega_g = 14.5400$ $\zeta_g = 0.30000$	<b>0.060813</b> $\omega_g = 14.7564$ $\zeta_g = 0.30000$	<b>0.060737</b> $\omega_g = 14.7542$ $\zeta_g = 0.30000$
$C_4$	<b>0</b>	<b>0.175165</b>	<b>0.196315</b>	<b>0.116033</b>	<b>0.514544</b>	<b>0.190882</b>	<b>0.145384</b>	<b>0.200995</b>	<b>0</b>	<b>0.947226</b>	<b>0.951560</b>
$C_5$	<b>0.905539</b>	<b>0.804580</b>	<b>0.705785</b>	<b>0.431867</b>	<b>2.824588</b>	<b>0.722743</b>	<b>0.574621</b>	<b>0.774253</b>	<b>1.925718</b>	<b>0.987402</b>	<b>0.963263</b>
$C_6$	<b>0.027950</b>	<b>0.224130</b>	<b>0.225663</b>	<b>0.131725</b>	<b>0.582699</b>	<b>0.216185</b>	<b>0.181141</b>	<b>0.231494</b>	<b>0.020030</b>	<b>0.120719</b>	<b>0.122831</b>

Figure 7. Control weight tradeoffs for the base  $H_2$  controller in the AMD systemFigure 8.  $H_2^{\text{Str}}/H_2^{\text{Act}}$  tradeoffs between RMS structural and actuator responses for the AMD system

controller resulting in the least mean square of the performance and constraint values (i.e.  $\text{argmin}_{\mathbf{K}} \sum_{i=1}^{10} J_i^2(\mathbf{K}) + \sum_{i=1}^6 C_i^2(\mathbf{K})$ ) is given in column #7 of Table IV; this controller was one that significantly decreased the peak active mass acceleration.

Figure 9. Pareto optimal curve for  $H_2/H_\infty$  AMD controllersFigure 10. Linear performance and constraint values for (near-) Pareto  $H_2/H_\infty$  optimal AMD controllers

If the robust performance component of the  $H_\infty$  constraint is removed, the resulting system has the zero controller as the  $H_\infty$  optimal robust controller. This complicates the characterization of the Pareto optimal tradeoff between the robust stability sensitivity, denoted here with  $(\cdot)^{RS}$ , and the  $H_2$  RMS response. Nevertheless, a number of mixed  $H_2/H_\infty^{RS}$  controllers were found as shown in Figure 12. The circled large

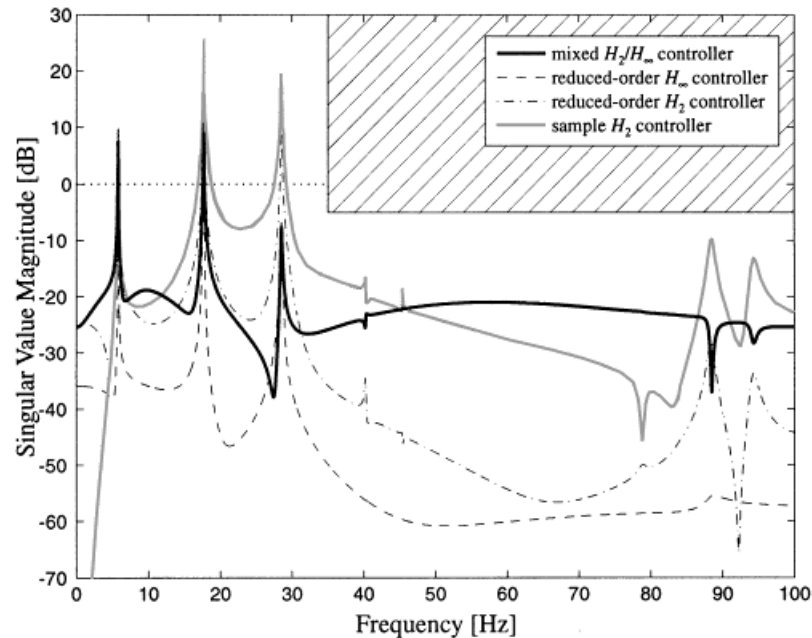


Figure 11. Input loop a gain for a reduced-order  $H_2/H_\infty$  optimal controller for the AMD system

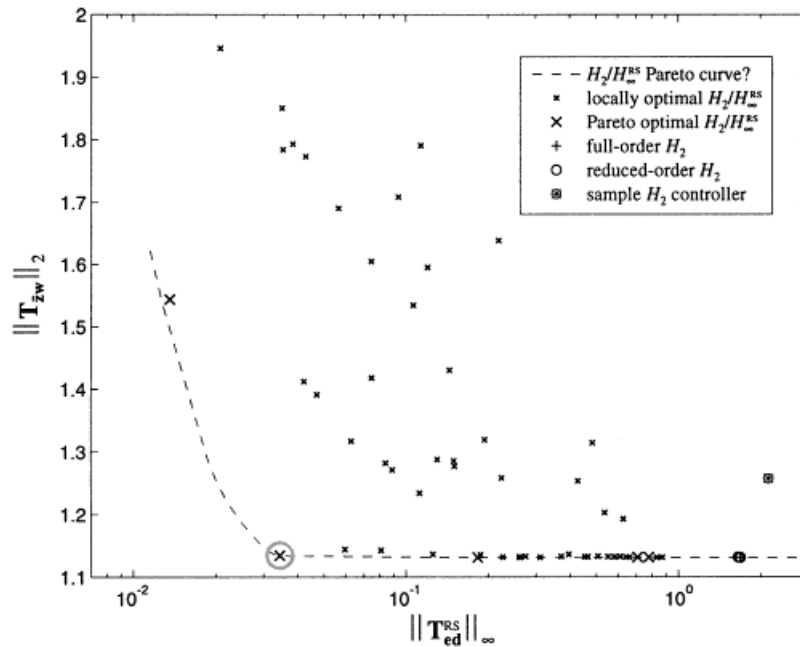


Figure 12.  $H_2/H_2^{RS}$  RMS performance and robust stability sensitivity for the AMD system

× represents a controller that provides a closed-loop RMS output only 0.28% larger than the reduced-order  $H_2$  optimal controller but 50 times as robust. The benchmark performance of this controller is given in column #8 of Table IV.



### Mixed $H_2/l_1$ design

The  $H_2/l_1$  optimization problem is more difficult than the  $H_2/H_\infty$  for several reasons. First, computing the  $l_1$  norm and its gradients with respect to controller parameters is quite expensive, limiting the number of trials that could be tested. The second is the lack of a good starting point; in the  $H_2/H_\infty$  design, it was observed that the optimization algorithm more readily relaxed constraints than restrict them, so using the  $H_\infty$  optimal controller as the starting point worked well. Here, the  $l_1$  optimal controller was unknown; several semianalytical methods, including those of Khammash<sup>12</sup>, were applied to solve for the  $l_1$  optimal controller, but the combination of very low damping in some modes and high frequency actuator dynamics prevented a solution. Consequently, the optimization software was modified to do unconstrained  $l_1$  optimization, which facilitated finding a number of controllers with smaller closed-loop  $l_1$  norms (but with no knowledge of the proximity to the  $l_1$  optimal controller).

Two different mixed  $H_2/l_1$  strategies were attempted to determine if one or the other had an impact on the peak performance and constraint measures defined above. The first used the  $l_1$  constraint with a constant weight on the input. About 40 mixed  $H_2/l_1$  designs were computed and are shown in Figure 13. The  $l_1$  norm was reduced somewhat from the  $H_2$  optimal controller, but not nearly as significantly as the  $H_\infty$  norm above. Some of these designs lowered the  $l_1$  norm of the closed loop system by 10% with less than 0.01 per cent increase in the RMS response. The Pareto curve is poorly defined at the low  $l_1$ -norm end due to difficulties with local minima. The large circled  $\times$  is the controller that obtained the smallest closed-loop  $l_1$  norm.

The second set of mixed  $H_2/l_1$  designs used a frequency-dependent input weight, that of the Kanai–Tajimi spectrum (with the ground frequency and damping that caused the worst response with the zero controller), to more closely model an earthquake record; to denote the frequency-dependent weight, these designs will be called  $H_2/l_1^{KT}$ . The results of the 55  $H_2/l_1^{KT}$  designs are plotted in Figure 14. The Pareto curve is a little better

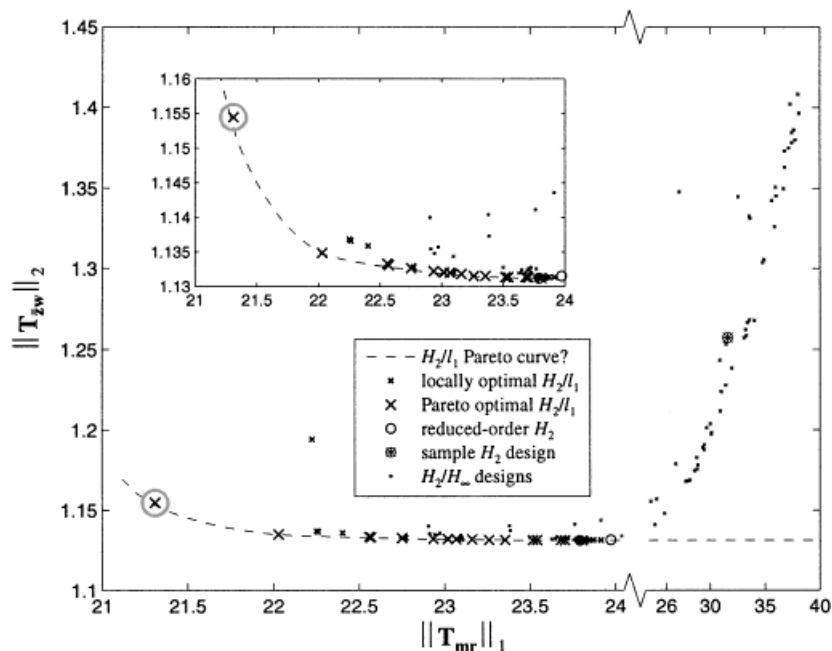


Figure 13. Mixed  $H_2/l_1$  optimal controllers for the AMD system (no frequency dependent weight)

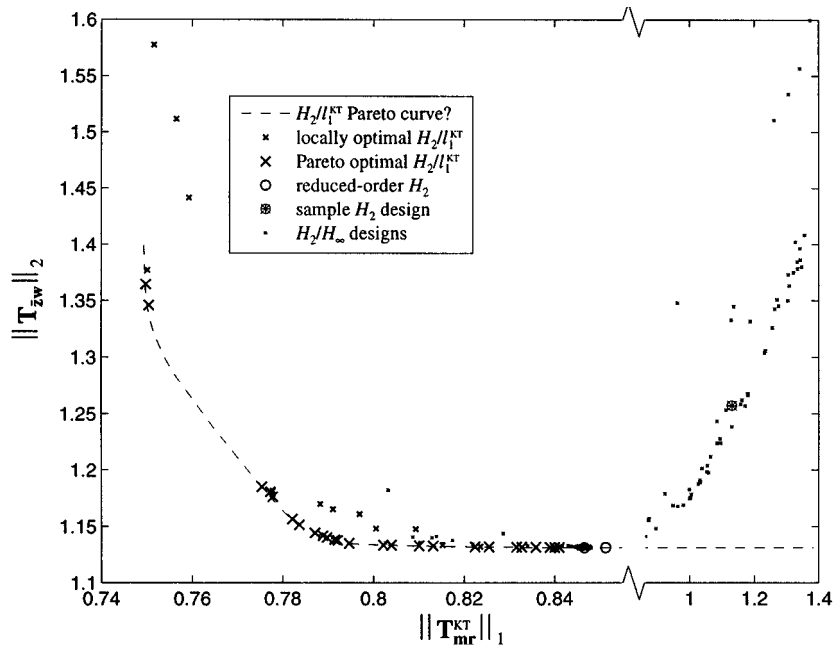


Figure 14. Mixed  $H_2/l_1^{KT}$  optimal controllers for the AMD system (Kanai–Tajimi frequency-dependent weight)

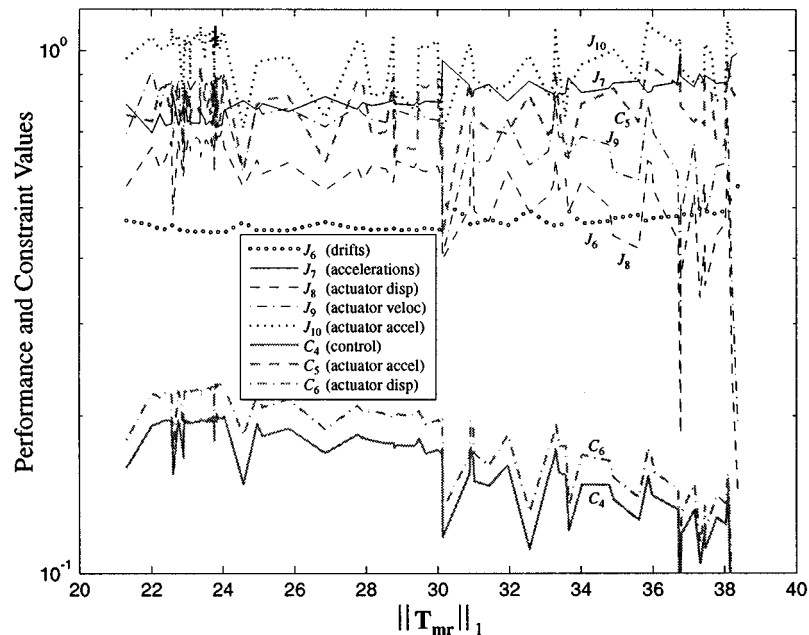


Figure 15. Peak linear performance and constraints are uncorrelated with closed-loop  $l_1$  norm

defined here, but without an optimal  $l_1$  controller to give the lower bound, it is impossible to define the curve with much certainty. The results here are similar in that 5–10 per cent drops in closed-loop  $l_1$  norm can be achieved with only nominal sacrifice in  $H_2$  performance.

Unfortunately, little correlation was found in the end between the closed-loop  $l_1$  norm and the values of the peak performance  $J_{6-10}$  and constraint  $C_{4-6}$  values, as shown for the non-frequency-dependent  $H_2/l_1$  designs in Figure 15 over a range of the  $l_1$  norm. The irregularity of the lines is to be expected, but the obvious lack of a general proportional trend between the  $l_1$  norm and the performance/constraint values shows that the  $l_1$  norm of the closed-loop cannot be used to predict peak responses to the known earthquake records.

The lack of correlation was rather surprising at first, but, in retrospect, it makes some sense. The  $l_1$  norm is the worst-case peak output over a very wide class of input excitations, those that are magnitude bounded (i.e. those signals in  $l_\infty$ ), which includes persistent signals that have DC offsets and long-term non-decaying behavior. The earthquake records used to evaluate the peak response of the closed-loop system for the benchmark, however, are finite in duration and more properly belong to the class of energy bounded signals ( $l_2$ ). The worst-case output magnitude over all possible energy-bounded inputs is governed by the  $H_2$  norm of the system. Thus, the  $H_2$  norm should be a better measure of peak output magnitudes than the  $l_1$  norm for the excitations of interest here.

### CONTROLLER DESIGNS: ACTIVE TENDON SYSTEM

#### *Performance with zero controller*

The performance of the active tendon system with the zero controller is given in column #9 of Table IV. This system has significantly tighter constraints, primarily on the actuator command voltage  $C_4$  and actuator force  $C_5$ .

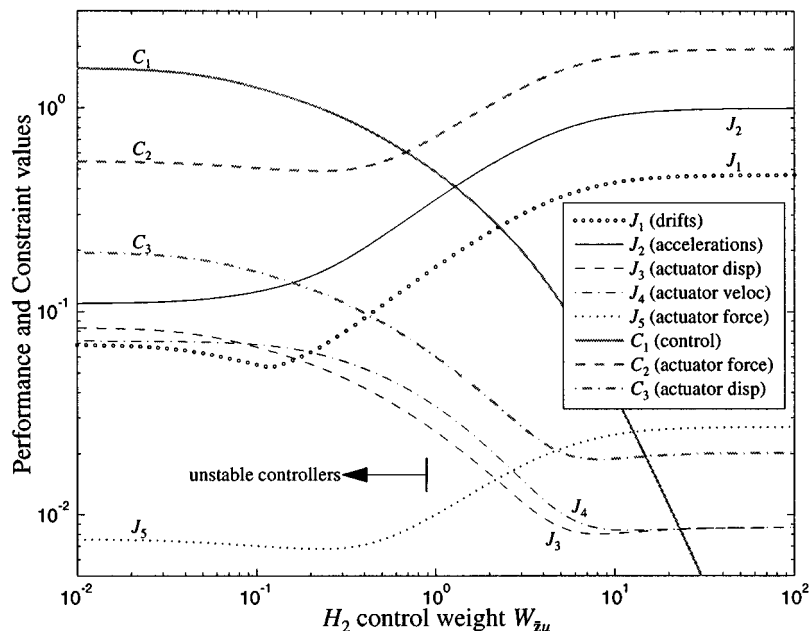


Figure 16. Control weight tradeoffs for the base  $H_2$  controller in the TEN system

### Base $H_2$ (LQG) design

The analysis parallels that of the AMD system. The base  $H_2$  tradeoff between performance and control effort is shown in Figure 16. The control effort weight  $W_{zu}$  has a very narrow admissible range since setting it to larger than 1.874 results in systems that violate the constraint on the actuator force ( $C_2 > 1$ ), and for  $W_{zu} < 0.887$ , the resulting controller is unstable. Thus, the weight chosen here is again unity.

Several initial  $H_2$  controllers, first full-order, then reduced order, were computed as described for the AMD system above; the resulting benchmark performance data for the best reduced-order  $H_2$  design is given in column #10 of Table IV. It must be noted that the actuator velocity here reached its peak in a very different location in  $(\omega_g, \zeta_g)$  space than other RMS performance and constraint values. A reduced-order  $H_\infty$  optimal controller were also used as initial conditions for mixed optimization.

### Mixed $H_2/H_\infty$ design

About 100 mixed  $H_2/H_\infty$  designs were computed to study the performance and robustness tradeoffs of the TEN system. The resulting norms are shown in Figure 17; as was previously noted, the optimization algorithm does get stuck in local minima for many of these designs, but some are very good at improving robustness with little loss of the  $H_2$  benefits. In fact, the  $H_2$  optimal controllers have terrible robustness characteristics for the active tendon system. The circled large  $\times$ 's are controllers that generate closed-loop plants with 2-norms 1.2 and 0.1 percent larger than the  $H_2$  optimal controller, but with infinity norms about 1/850th and 1/175th that of the  $H_2$  controller. The input loop gains for these two designs are shown in Figure 18 and the performance evaluation of the second is given in column #11 of Table IV. The optimal  $H_2$  controller results in a fair sensitivity to high-frequency modelling errors that could, at worst, cause

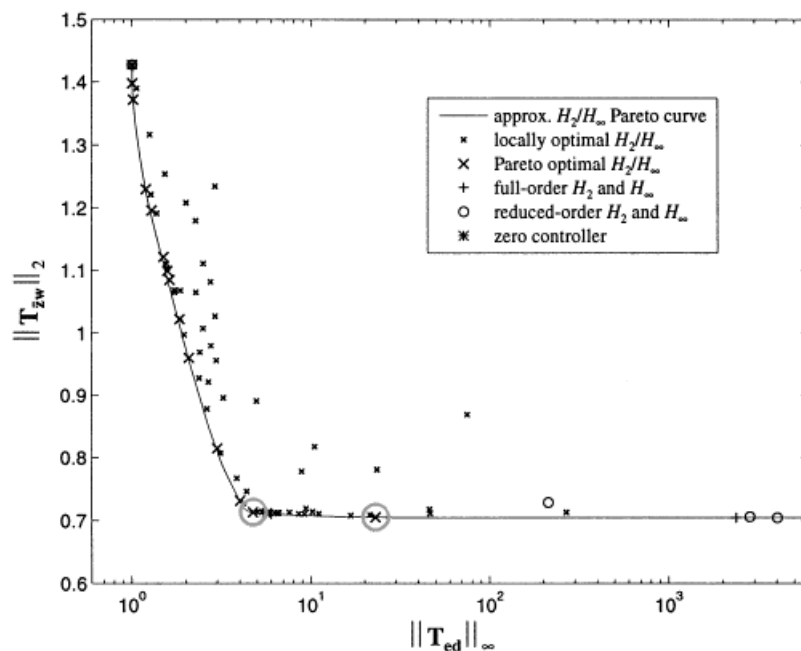


Figure 17. Pareto optimal curve for  $H_2/H_\infty$  TEN controllers

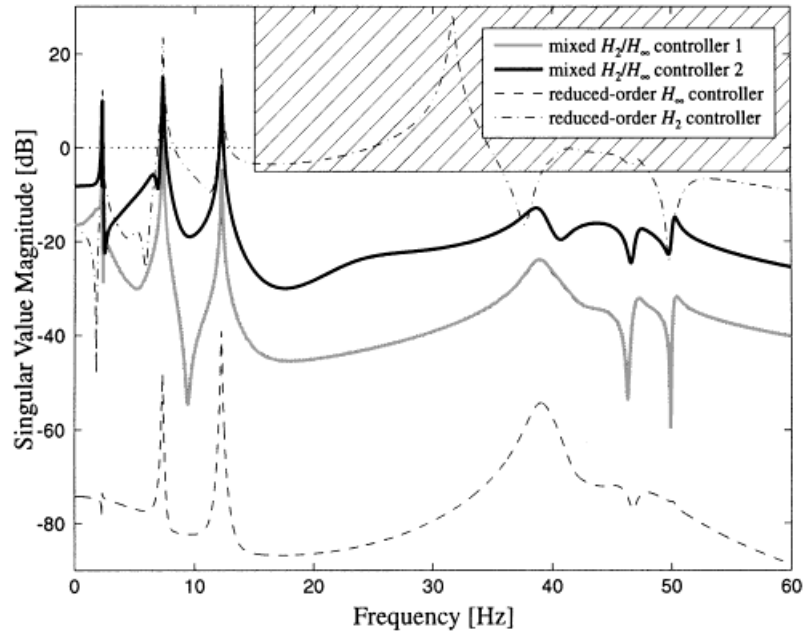


Figure 18. Input loop gain for reduced-order  $H_2/H_\infty$  optimal controller for the TEN system

instability if the truncated high-frequency modes dominate under some loading conditions. The mixed  $H_2/H_\infty$  designs, however, while retaining most of the RMS performance of the  $H_2$  design, also capture some of the robustness of the  $H_\infty$  optimal controller.

#### *Mixed $H_2/l_1$ design*

Due to the computational intensity and its inability to suppress peak responses of the AMD system, no mixed  $H_2/l_1$  optimization was done on the active tendon system.

### OBSERVATIONS AND CONCLUDING REMARKS

The mixed  $H_2/H_\infty$  optimal control design was able to provide controllers with significantly greater robustness characteristics at little cost to RMS and peak performance measures defined in the benchmark. This was especially noticeable in the active tendon system, where the infinity-norms of the  $H_2$  optimal controllers were quite large. The robustness was a relatively minor point in the benchmark definition, but it is a critical safety issue for actively controlled buildings and must be included in an evaluation of controller design.

The performance of the controllers mentioned herein is generally better than the sample controller provided with the benchmark, though the definition of 'better' is, of course, rather application specific and will vary with the requirements of the structure. The 50 'best' AMD results here averaged about 20 per cent less than the sample controller (where all 10 performance measures and the 6 weighted constraints are averaged together), with the best being 28 per cent less.

The computer resources required for the multiobjective optimizations performed for this study vary significantly depending on what objective and constraint functions are used. The  $l_1$  norms were routinely significantly more computationally intensive than the others, up to the better part of a day of CPU time on

an HP9000/780 workstation. Other mixed-norm designs, such as the RMS structural/actuator response tradeoffs, generally took only a few minutes on the same platform. A full Simulink<sup>®</sup> simulation (up to 300 sec) of one Kanai–Tajimi spectrum for one controller took about 105 minutes, so the evaluation was at least as expensive as the design process.

It was observed in the process of doing  $H_2/H_\infty$  mixed designs that the optimization algorithm more readily relaxed constraints than restricted them, being less likely to get stuck at local minima. For example, in an  $H_2/H_\infty$  problem, starting with an  $H_\infty$  optimal controller and relaxing the constraint often results in solutions that are closer to the Pareto optimal curve than starting with a controller that is closer to  $H_2$  optimal and searching for a more constrained  $H_\infty$  value.

It may be noted that the  $H_\infty$  portion of the problem tends toward the conservative side since the overall uncertainty is structured due to the two separate uncertainty blocks. Walker and Ridgely<sup>16</sup> have done some work in mixed  $H_2/\mu$  optimization, so the above development could replace the  $H_\infty$  constraint with a  $\mu$  constraint to more closely model the true system uncertainty.

Investigation into  $l_1$  optimization in the mixed-norm context for this benchmark proved inadequate for reducing peak response and accommodating peak hardware constraints. Since an earthquake is generally an energy-bounded excitation (it dies out after a short time), the  $l_1$  norm might not be the best induced norm to use since that assumes the input to be in too broad a class of signals ( $l_\infty$ ). (In fact, it may be argued that an earthquake excitation is in  $l_1$ , an even narrower class of signals whose  $l_1$  norm is bounded.) The  $H_2$  norm, already included for minimizing RMS response, has a bigger impact on peak outputs to the real earthquake records than did controllers with small corresponding  $l_1$  norms.

This study demonstrates the usefulness of mixed-norm optimization to allow the control designer to dramatically improve robustness while retaining the performance of  $H_2$ /LQG optimal controllers. Understanding these tradeoffs is essential to the control design process.

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